Bending Stress

- In engineering practice, the machine parts of structural members may be subjected to static or dynamic loads which cause bending stress in the sections besides other types of stresses such as tensile, compressive and shearing stresses.

- A little consideration will show that when a beam is subjected to the bending moment, the fibres on the upper side of the beam will be shortened due to compression and those on the lower side will be elongated due to tension.
It may be seen that somewhere between the top and bottom fibres there is a surface at which the fibres are neither shortened nor lengthened. Such a surface is called neutral surface. The intersection of the neutral surface with any normal cross-section of the beam is known as neutral axis. The stress distribution of a beam is shown in figure.
The bending equation is given by

\[
\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}
\]

where

\( M \) = Bending moment acting at the given section,

\( \sigma \) = Bending stress,

\( I \) = Moment of inertia of the cross-section about the neutral axis,

\( y \) = Distance from the neutral axis to the extreme fibre,

\( E \) = Young’s modulus of the material of the beam, and

\( R \) = Radius of curvature of the beam.
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<th>(A)</th>
<th>(I)</th>
<th>(y)</th>
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<td>( \frac{\pi}{4} \times d^2 )</td>
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<td>( d \over 2 )</td>
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<td>8. Hollow circle</td>
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<td>( I_{xx} = I_{yy} = \frac{\pi}{64} (d^4 - d_1^4) )</td>
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<td>( I_{xx} = \frac{\pi}{4} \times a^3 b )</td>
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<td>*Distance from the neutral axis to the extreme fibre ((y))</td>
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<td></td>
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<td>Section</td>
<td>((A))</td>
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<td>(I_{xx} = \frac{\pi}{4} (b a^3 - b_1 a_1^3))</td>
<td>(a)</td>
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<td></td>
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<tr>
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<td>(b h - b_1 h_1)</td>
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<td>(h = H - h_1)</td>
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<td></td>
<td></td>
<td>(= \frac{a H^2 + b t^2}{2 (a H + b t)})</td>
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Example:

pump lever rocking shaft is shown in figure. The pump lever exerts forces of 25 kN and 35 kN concentrated at 150 mm and 200 mm from the left and right hand bearing respectively.

Find the diameter of the central portion of the shaft, if the stress is not to exceed 100 MPa.
Solution.

Given : \( \sigma_b = 100 \text{ MPa} = 100 \text{ N/mm}^2 \)

Let \( RA \) and \( RB \) = Reactions at \( A \) and \( B \) respectively.

Taking moments about \( A \), we have

\[
RB \times 950 = 35 \times 750 + 25 \times 150 = 30000
\]

\[
\therefore RB = 30000 / 950 = 31.58 \text{ kN} = 31.58 \times 10^3 \text{ N}
\]

and \( RA = (25 + 35) - 31.58 = 28.42 \text{ kN} = 28.42 \times 10^3 \text{ N} \)

\[
\therefore \text{ Bending moment at } C
\]

\[
= RA \times 150 = 28.42 \times 10^3 \times 150 = 4.263 \times 10^6 \text{ N-mm}
\]

and bending moment at \( D = RB \times 200 = 31.58 \times 10^3 \times 200 = 6.316 \times 10^6 \text{ N-mm} \)
We see that the maximum bending moment is at \( D \), therefore maximum bending moment, \( M \)

\[ M = 6.316 \times 10^6 \text{ N-mm}. \]

Let

\[ d = \text{Diameter of the shaft}. \]

\[ I = \pi \cdot d^4 / 64 \]

We know that bending stress \((\sigma_b)\),

\[ 100 = M \cdot Y / I \]

\[ d = 86.3 \text{ mm} \quad \text{say 90 mm} \]
Example:
A beam of uniform rectangular cross-section is fixed at one end and carries an electric motor weighing 400 N at a distance of 300 mm from the fixed end. The maximum bending stress in the beam is 40 MPa. Find the width and depth of the beam, if depth is twice that of width.
Solution.

Given: \(W = 400 \text{ N} \); \(L = 300 \text{ mm} \);

\(\sigma_b = 40 \text{ MPa} = 40 \text{ N/mm}^2 \); \(h = 2b\)

The beam is shown in the figure

Let \(b\) = Width of the beam in mm,

And \(h\) = Depth of the beam in mm.

\[I = \frac{bh^3}{12} = b(2b)^3 / 12\]

Maximum bending moment (at the fixed end),

\[M = W\cdot L = 400 \times 300 = 120 \times 10^3 \text{ N-mm}\]

We know that bending stress \((\sigma_b)\),

\[40 = \frac{MY}{I}\]

\(b = 16.5 \text{ mm}\), \(h = 2b = 2 \times 16.5 = 33 \text{ mm}\)
Example

A steel bar of $20 \times 60$-mm rectangular cross section is subjected to two equal and opposite couples acting in the vertical plane of symmetry of the bar. Determine the value of the bending moment $M$ that causes the bar to yield. Assume $\sigma_Y = 250$ MPa.
Solution:

Since the neutral axis must pass through the centroid $C$ of the cross section, we have $c = 30$ mm. On the other hand, the centroidal moment of inertia of the rectangular cross section is

$$I = \frac{1}{12} bh^3 = \frac{1}{12} (20 \text{ mm})(60 \text{ mm})^3 = 360 \times 10^3 \text{ mm}^4$$

Solving Eq. (3-15) for $M$, and substituting the above data, we have

$$M = \frac{I}{c} \sigma_m = \frac{360 \times 10^{-9} \text{ m}^4}{0.03 \text{ m}} (250 \text{ MPa})$$

$$M = 3 \text{ kN} \cdot \text{m}$$
The rectangular tube shown is extruded from an aluminum alloy for which $\sigma_Y = 275$ MPa, $\sigma_U = 415$ MPa, and $E = 73$ GPa. Neglecting the effect of fillets, determine (a) the bending moment $M$ for which the factor of safety will be 3.00, (b) the corresponding radius of curvature of the tube.
SOLUTION

Moment of Inertia. Considering the cross-sectional area of the tube as the difference between the two rectangles shown and recalling the formula for the centroidal moment of inertia of a rectangle, we write

$$I = \frac{1}{12}(80 \text{ mm})(120 \text{ mm})^3 - \frac{1}{12}(68 \text{ mm})(108 \text{ mm})^3 \quad I = 4.38 \times 10^6 \text{ mm}^4$$

Allowable Stress. For a factor of safety of 3.00 and an ultimate stress of 415 MPa, we have

$$\sigma_{\text{all}} = \frac{\sigma_U}{F.S.} = \frac{415 \text{ MPa}}{3.00} = 138 \text{ MPa}$$
a. Bending Moment. With $c = \frac{1}{2}(120 \text{ mm}) = 60 \text{ mm}$, we write

$$\sigma_{\text{all}} = \frac{Mc}{I} \quad M = \frac{I}{c} \sigma_{\text{all}} = \frac{4.38 \times 10^{-6} \text{ m}^4}{0.06 \text{ m}} (138 \text{ MPa}) \quad M = 10.1 \text{ kN} \cdot \text{m}$$

b. Radius of Curvature. Recalling that $E = 73$ GPa we substitute this value and the values obtained for $I$ and $M$ into Eq. 3-21 and find

$$\frac{1}{\rho} = \frac{M}{EI} = \frac{10.1 \text{ kN} \cdot \text{m}}{(73 \text{ GPa})(4.38 \times 10^{-6} \text{ m}^4)} = 0.0316 \text{ m}^{-1}$$

$$\rho = 31.7 \text{ m}$$
An open-link chain is obtained by bending low-carbon steel rods of 12-mm diameter into the shape shown in Figure. Knowing that the chain carries a load of 700 N, determine (a) the largest tensile and compressive stresses in the straight portion of a link, (b) the distance between the centroidal and the neutral axis of a cross section.
Solution:

(a) Largest Tensile and Compressive Stresses.
The internal forces in the cross section are equivalent to a centric force $P$ and a bending couple $M$

$$P = 700 \text{ N}$$

$$M = Pd = (700 \text{ N})(0.016 \text{ m}) = 11.2 \text{ N} \cdot \text{m}$$
The corresponding stress distributions are shown in parts a and b of Fig. The distribution due to the centric force \( P \) is uniform and equal to \( \sigma_0 = P/A \). We have

\[
A = \pi c^2 = \pi (6 \text{ mm})^2 = 113.1 \text{ mm}^2
\]

\[
\sigma_0 = \frac{P}{A} = \frac{700 \text{ N}}{113.1 \text{ mm}^2} = 6.2 \text{ MPa}
\]
The distribution due to the bending couple $M$ is linear with a maximum stress $\sigma_m = Mc/I$. We write

$$I = \frac{1}{4} \pi c^4 = \frac{1}{4} \pi (6 \text{ mm})^4 = 1017.9 \text{ mm}^4$$

$$\sigma_m = \frac{Mc}{I} = \frac{(11.2 \times 10^3 \text{ N} \cdot \text{ mm})(6 \text{ mm})}{1017.9 \text{ mm}^4} = 66 \text{ MPa}$$

Superposing the two distributions, we obtain the stress distribution corresponding to the given eccentric loading (Fig. C). The largest tensile and compressive stresses in the section are found to be, respectively,

$$\sigma_t = \sigma_0 + \sigma_m = 6.2 + 66 = 72.2 \text{ MPa}$$

$$\sigma_c = \sigma_0 - \sigma_m = 6.2 - 66 = -59.8 \text{ MPa}$$
INTRODUCTION:

In this section structural members and machine parts that are in torsion will be considered. We will analyze the stresses and strains in members of circular cross section subjected to twisting couples, or torques, \( T \) and \( T' \). These couples have a common magnitude \( T \), and opposite senses. They are vector quantities and can be represented either by curved arrows as in Figure (a), or by couple vectors as in Figure (b).
The most common application is provided by transmission shafts, which are used to transmit power from one point to another. For example, the shaft shown in next Figures:

In the automotive power train shown, the shaft transmits power from the engine to the rear wheels.
When a machine member is subjected to the action of two equal and opposite couples acting in parallel planes (or torque or twisting moment), then the machine member is said to be subjected to **torsion**. The stress set up by torsion is known as **torsional shear stress**. It is zero at the centroidal axis and maximum at the outer surface.

Consider a shaft fixed at one end and subjected to a torque \( (T) \) at the other end as shown in figure. As a result of this torque, every cross-section of the shaft is subjected to torsional shear stress. We have discussed above that the torsional shear stress is zero at the centroidal axis and maximum at the outer surface.
The maximum torsional shear stress at the outer surface of the shaft may be obtained from the following equation:

\[
\tau = \frac{T}{r} = \frac{C}{J} \frac{\theta}{l}
\]

\(\tau\) = Torsional shear stress induced at the outer surface of the shaft or maximum shear stress,

\(r\) = Radius of the shaft,

\(T\) = Torque or twisting moment,

\(J\) = Second moment of area of the section about its polar axis or polar moment of inertia,

\(C\) = Modulus of rigidity for the shaft material,

\(l\) = Length of the shaft, and

\(\theta\) = Angle of twist in radians on a length \(l\).
Example:

A shaft is transmitting 100 kW at 160 r.p.m. Find a suitable diameter for the shaft, if the maximum torque transmitted exceeds the mean by 25%. Take maximum allowable shear stress as 70 MPa.

Solution. Given: \( P = 100 \text{ kW} = 100 \times 10^3 \text{ W} \); \( N = 160 \text{ r.p.m.} \); \( T_{max} = 1.25 T_{\text{mean}} \); \( \tau = 70 \text{ MPa} \)

\[
= 70 \text{ N/mm}^2
\]

Let \( T_{\text{mean}} \) = Mean torque transmitted by the shaft in N-m, and \( d \) = Diameter of the shaft in mm.

We know that the power transmitted \( (P) \),

\[
100 \times 10^3 = \frac{2 \pi N \cdot T_{\text{mean}}}{60} = \frac{2\pi \times 160 \times T_{\text{mean}}}{60} = 16.76 T_{\text{mean}}
\]

\[
\therefore \quad T_{\text{mean}} = \frac{100 \times 10^3}{16.76} = 5966.6 \text{ N-m}
\]
and maximum torque transmitted,

\[ T_{\text{max}} = 1.25 \times 5966.6 = 7458 \text{ N-m} = 7458 \times 10^3 \text{ N-mm} \]

We know that maximum torque \( T_{\text{max}} \),

\[ 7458 \times 10^3 = \frac{\pi}{16} \times \tau \times d^3 = \frac{\pi}{16} \times 70 \times d^3 = 13.75 \, d^3 \]

\[ \therefore \quad d^3 = 7458 \times 10^3 / 13.75 = 542.4 \times 10^3 \text{ or } d = 81.5 \text{ mm} \]
Example:

A steel shaft 35 mm in diameter and 1.2 m long held rigidly at one end has a hand wheel 500 mm in diameter keyed to the other end. The modulus of rigidity of steel is 80 GPa.

1. What load applied to tangent to the rim of the wheel produce a torsional shear of 60 MPa?

2. How many degrees will the wheel turn when this load is applied?
Solution. Given: \( d = 35 \text{ mm} \) or \( r = 17.5 \text{ mm} \); \( l = 1.2 \text{ m} = 1200 \text{ mm} \); \( D = 500 \text{ mm} \) or \( R = 250 \text{ mm} \); \( C = 80 \text{ GPa} = 80 \text{ kN/mm}^2 = 80 \times 10^3 \text{ N/mm}^2 \); \( \tau = 60 \text{ MPa} = 60 \text{ N/mm}^2 \)

1. Load applied to the tangent to the rim of the wheel

Let \( W = \) Load applied (in newton) to tangent to the rim of the wheel.

We know that torque applied to the hand wheel,

\[
T = W \cdot R = W \times 250 = 250 W \text{ N-mm}
\]

and polar moment of inertia of the shaft,

\[
J = \frac{\pi}{32} \times d^4 = \frac{\pi}{32} (35)^4 = 147.34 \times 10^3 \text{ mm}^4
\]

We know that

\[
\frac{T}{J} = \frac{\tau}{r}
\]

\[
\therefore \quad \frac{250 W}{147.34 \times 10^3} = \frac{60}{17.5} \quad \text{or} \quad W = \frac{60 \times 147.34 \times 10^3}{17.5 \times 250} = 2020 \text{ N} \quad \text{Ans.}
\]
2. Number of degrees which the wheel will turn when load \( W = 2020 \, N \) is applied

Let \( \theta = \text{Required number of degrees.} \)

We know that \( \frac{T}{J} = \frac{C \cdot \theta}{l} \)

\[ \therefore \quad \theta = \frac{T \cdot l}{C \cdot J} = \frac{250 \times 2020 \times 1200}{80 \times 10^3 \times 147.34 \times 10^3} = 0.05^\circ \quad \text{Ans.} \]
Example:

The horizontal shaft $AD$ is attached to a fixed base at $D$ and is subjected to the torques shown. A 44-mm-diameter hole has been drilled into portion $CD$ of the shaft. Knowing that the entire shaft is made of steel for which $G = 77$ GPa, determine the angle of twist at end $A$. 
Solution:

Since the shaft consists of three portions $AB$, $BC$, and $CD$, each of uniform cross section and each with a constant internal torque, Eq. may be used.

**Statics.** Passing a section through the shaft between $A$ and $B$ and using the free body shown, we find

\[ \Sigma M_x = 0: \quad (250 \text{ N} \cdot \text{m}) - T_{AB} = 0 \quad T_{AB} = 250 \text{ N} \cdot \text{m} \]

Passing now a section between $B$ and $C$, we have

\[ \Sigma M_x = 0: \quad (250 \text{ N} \cdot \text{m}) + (2000 \text{ N} \cdot \text{m}) - T_{BC} = 0 \quad T_{BC} = 2250 \text{ N} \cdot \text{m} \]

Since no torque is applied at $C$,

\[ T_{CD} = T_{BC} = 2250 \text{ N} \cdot \text{m} \]
**Polar Moments of Inertia**

\[ J_{AB} = \frac{\pi}{2} c^4 = \frac{\pi}{2} (0.015 \text{ m})^4 = 0.0795 \times 10^{-6} \text{ m}^4 \]

\[ J_{BC} = \frac{\pi}{2} c^4 = \frac{\pi}{2} (0.030 \text{ m})^4 = 1.272 \times 10^{-6} \text{ m}^4 \]

\[ J_{CD} = \frac{\pi}{2} (c_2^4 - c_1^4) = \frac{\pi}{2} [(0.030 \text{ m})^4 - (0.022 \text{ m})^4] = 0.904 \times 10^{-6} \text{ m}^4 \]
Angle of Twist. Using Eq. and recalling that $G = 77$ GPa for the entire shaft, we have

$$\phi_A = \sum_i \frac{T_i L_i}{J_i G} = \frac{1}{G} \left( \frac{T_{AB} L_{AB}}{J_{AB}} + \frac{T_{BC} L_{BC}}{J_{BC}} + \frac{T_{CD} L_{CD}}{J_{CD}} \right)$$

$$\phi_A = \frac{1}{77 \text{ GPa}} \left[ \frac{(250 \text{ N } \cdot \text{ m})(0.4 \text{ m})}{0.0795 \times 10^{-6} \text{ m}^4} + \frac{(2250)(0.2)}{1.272 \times 10^{-6}} + \frac{(2250)(0.6)}{0.904 \times 10^{-6}} \right]$$

$$= 0.01634 + 0.00459 + 0.01939 = 0.0403 \text{ rad}$$

$$\phi_A = (0.0403 \text{ rad}) \frac{360^\circ}{2\pi \text{ rad}} = 2.31^\circ$$